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Strong convergence rates for backward Euler on a class of nonlinear jump-diffusion problems

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Abstract

We generalise the current theory of optimal strong convergence rates for implicit Euler-based methods by allowing for Poisson-driven jumps in a stochastic differential equation (SDE). More precisely, we show that under one-sided Lipschitz and polynomial growth conditions on the drift coefficient and global Lipschitz conditions on the diffusion and jump coefficients, three variants of backward Euler converge with strong order of one half. The analysis exploits a relation between the backward and explicit Euler methods.
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1. Introduction

In this work we look at Itô stochastic differential equations (SDEs) with Poisson-driven jumps. More precisely, we focus on SDEs of the form

$$\mathrm{d}X(t) = f(X(t^-))\,\mathrm{d}t + g(X(t^-))\,\mathrm{d}W(t) + h(X(t^-))\,\mathrm{d}N(t), \qquad X(0^-) = X_0, \tag{1}$$

over a finite time interval $[0, T]$. Here, $X(t^-)$ denotes $\lim_{s\to t^-} X(s)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^n$, $W(t)$ is an m-dimensional Brownian motion and $N(t)$ is a scalar Poisson process with intensity λ. Problems of this form arise in many areas of science [3,12] and, perhaps most significantly, in mathematical finance [1,4].

Strong convergence of fixed timestep methods for jump-SDEs has been considered in [2,9–11] in the case of explicit methods and [5,6] in the case of implicit methods. It is proved in [5,6] that, as with deterministic problems, implicit methods offer benefits in terms of linear and nonlinear stability. Further, Higham and Kloeden [5] show that strong convergence results for implicit methods can be derived for classes of nonlinear problems that do not satisfy a global Lipschitz condition.

Our aim now is to show that by imposing a further, polynomial-like condition on the drift, optimal strong convergence rates can be established for three implicit methods based on backward Euler. This order is optimal in the sense that

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the same order arises for non-jump-SDEs under global Lipschitz conditions on f and g [8]. The analysis uses ideas from [7, Sections 4 and 5], where analogous results are derived in the non-jump case.

The next section states the assumptions that we impose on the problem, the most notable being a one-sided Lipschitz condition on the drift. Such a condition has been used successfully in many studies of numerical methods for evolutionary problems. In Section 3 we show that a basic, explicit Euler–Maruyama discretisation has an optimal strong convergence rate under the assumption that the numerical approximation has bounded moments. Then, in Section 4 we show that a split-step variant of backward Euler

- has bounded moments, and
- corresponds to the explicit Euler–Maruyama method applied to a slightly perturbed problem.

Using the result from Section 3, this allows us to prove optimal strong convergence for the implicit method. Building on this result, in Section 5 we show that a more conventional implementation of the backward Euler method also has optimal strong convergence order.

Overall, this work combines ideas from [5], where jump-SDEs are studied but *rates* of convergence are not considered and [7], where rates are proved for *non-jump*-SDEs. We have attempted to make the material as self-contained as possible, but refer to [5,7] for more detailed descriptions of some of the analytical techniques.

2. Conditions on the SDE

Throughout, we assume that the initial data have bounded moments, that is, for each $p>0$ there is a finite M_p such that

$$\mathbb{E}|X_0|^p < M_p. \tag{2}$$

We further assume that

$$f, g, h \in C^1, \tag{3}$$

the drift coefficient f satisfies a one-sided Lipschitz condition

$$\langle x-y, f(x)-f(y)\rangle \leqslant \mu |x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^n, \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product, and the diffusion and jump coefficients satisfy global Lipschitz conditions

$$|g(x)-g(y)|^2 \leqslant L_g |x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^n, \tag{5}$$

$$|h(x)-h(y)|^2 \leqslant L_h |x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^n, \tag{6}$$

where $|\cdot|$ denotes both the Euclidean vector norm and the Frobenius matrix norm.

Under these conditions it is shown in [5, Lemma 1] that (1) has a unique solution with all moments bounded. Further, in [5], strong convergence is established for implicit methods based on backward Euler. However, *rates* of convergence are not given. In this work we impose the extra condition that f behaves polynomially, in the sense that there is a constant D and a positive integer q for which

$$|f(x)-f(y)|^2 \leqslant D(1+|x|^q+|y|^q)|x-y|^2 \quad \text{for all } x, y \in \mathbb{R}^n, \tag{7}$$

and show that optimal rates can be recovered. This extra condition was used in [7], where non-jump SDEs were studied. In essence, inequality (7) makes it possible to exploit moment bounds on the numerical solution.

3. Euler–Maruyama

One generalisation of the Euler–Maruyma method [8] to the jump problem (1) has the form $Y_0 = X_0$ and

$$Y_{n+1} = Y_n + f(Y_n)\Delta t + g(Y_n)\Delta W_n + h(Y_n)\Delta N_n. \tag{8}$$

Here Δt is a fixed timestep, $\Delta W_n = W(t_{n+1}) - W(t_n)$ (with $t_n = n\Delta t$) is a Brownian increment, $\Delta N_n = N(t_{n+1}) - N(t_n)$ is a Poisson increment and $Y_n \approx X(t_n)$. Given a discrete-time approximation $\{Y_n\}_{n\geqslant 0}$, we define a continuous-time approximation $\overline{Y}(t)$ by

$$\overline{Y}(t) = X_0 + \int_0^t f(Y(s^-))\,\mathrm{d}s + \int_0^t g(Y(s^-))\,\mathrm{d}W(s) + \int_0^t h(Y(s^-))\,\mathrm{d}N(s), \tag{9}$$

where

$$Y(t) = Y_n \quad \text{for } t \in [t_n, t_{n+1}). \tag{10}$$

We remark that $\overline{Y}(t)$ is not computable, since it requires knowledge of the entire Brownian and Poisson paths, not just their Δt-increments. However, since $\overline{Y}(t_n) = Y_n$, an error bound for $\overline{Y}(t)$ will automatically imply an error bound for $\{Y_n\}_{n\geqslant 0}$.

The following result, which extends [7, Theorem 4.4], shows that this method is strongly convergent with order $\frac{1}{2}$ if the numerical solution has bounded moments.

Throughout this work, we use K to denote a generic constant (independent of Δt) that may change from line to line.

Theorem 1. *Under the assumptions* (2)–(7), *if*

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} |\overline{Y}(t)|^p < \infty \quad \textit{for all } p > 1,$$

then the continuous-time extension (9) *of the Euler–Maruyama method* (8) *satisfies*

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} |\overline{Y}(t) - X(t)|^2 = \mathrm{O}(\Delta t).$$

Proof. We must adapt the arguments in the proof of [7, Theorem 4.4]. This is because, unlike $W(t_{n+1}) - W(t_n)$, the Poisson increment $N(t_{n+1}) - N(t_n)$ has all moments of order $\mathrm{O}(\Delta t)$, and so an extension of [7, Lemma 4.3] is not possible.

Let $e(t) := X(t) - \overline{Y}(t)$. From the identity

$$X(t) = X_0 + \int_0^t f(X(s^-))\,\mathrm{d}s + \int_0^t g(X(s^-))\,\mathrm{d}W(s) + \int_0^t h(X(s^-))\,\mathrm{d}N(s)$$

and (9), we apply the Itô formula [2] to obtain

$$\begin{aligned}
|e(t)|^2 &= \int_0^t 2\langle f(X(s^-)) - f(Y(s^-)), e(s^-)\rangle + |g(X(s^-)) - g(Y(s^-))|^2\,\mathrm{d}s \\
&\quad + \int_0^t 2\langle e(s^-), (g(X(s^-)) - g(Y(s^-)))\,\mathrm{d}W(s)\rangle \\
&\quad + \int_0^t 2\langle e(s^-), h(X(s^-)) - h(Y(s^-))\rangle + |h(X(s^-)) - h(Y(s^-))|^2\,\mathrm{d}N(s) \\
&= \int_0^t (2\langle f(X(s^-)) - f(\overline{Y}(s^-)), e(s^-)\rangle \\
&\quad + 2\langle f(\overline{Y}(s^-)) - f(Y(s^-)), e(s^-)\rangle + |g(X(s^-)) - g(Y(s^-))|^2)\,\mathrm{d}s \\
&\quad + \int_0^t 2\langle e(s^-), (g(X(s^-)) - g(Y(s^-)))\,\mathrm{d}W(s)\rangle \\
&\quad + \int_0^t 2\langle e(s^-), h(X(s^-)) - h(Y(s^-))\rangle + |h(X(s^-)) - h(Y(s^-))|^2\,\mathrm{d}N(s).
\end{aligned}$$

Introducing the compensated Poisson process

$$\widetilde{N}(t) := N(t) - \lambda t, \tag{11}$$

which is a martingale, we have

$$\begin{aligned}|e(t)|^2 = \int_0^t & 2\langle f(X(s^-)) - f(\overline{Y}(s^-)), e(s^-)\rangle + 2\langle f(\overline{Y}(s^-)) - f(Y(s^-)), e(s^-)\rangle \\ &+ |g(X(s^-)) - g(Y(s^-))|^2 + 2\lambda\langle e(s^-), h(X(s^-)) - h(\overline{Y}(s^-))\rangle \\ &+ \lambda |h(X(s^-)) - h(\overline{Y}(s^-))|^2 \,\mathrm{d}s + M(t),\end{aligned}$$

where $M(t)$ is a martingale. Using the Lipschitz and growth conditions (4)–(7), this gives

$$\begin{aligned}|e(t)|^2 &\leqslant K \int_0^t |e(s^-)|^2 \,\mathrm{d}s + K \int_0^t (1 + |Y(s^-)|^q + |\overline{Y}(s^-)|^q)|Y(s^-) - \overline{Y}(s^-)|^2 \,\mathrm{d}s + M(t) \\ &\leqslant K \int_0^t |e(s^-)|^2 \,\mathrm{d}s + K \left(\sup_{0\leqslant s\leqslant t} |Y(s) - \overline{Y}(s)|^2\right) \int_0^t 1 + |Y(s^-)|^q + |\overline{Y}(s^-)|^q \,\mathrm{d}s + M(t).\end{aligned} \tag{12}$$

Now, for $t \in [k\Delta t, (k+1)\Delta t)$,

$$\begin{aligned}|Y(t) - \overline{Y}(t)|^2 &= |(t - t_k) f(Y_k) + g(Y_k)(W(t) - W(t_k)) + h(Y_k)(N(t) - N(t_k))|^2 \\ &\leqslant 3(\Delta t^2 |f(Y_k)|^2 + |g(Y_k)|^2 |W(t) - W(t_k)|^2 + |h(Y_k)|^2 |N(t) - N(t_k)|^2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E} \sup_{t_k \leqslant t < t_{k+1}} |Y(s) - \overline{Y}(s)|^2 &\leqslant K\Delta t^2 + K\mathbb{E} \sup_{t_k \leqslant t < t_{k+1}} |W(t) - W(t_k)|^2 + K\mathbb{E} \sup_{t_k \leqslant t < t_{k+1}} |N(t) - N(t_k)|^2 \\ &\leqslant K\Delta t.\end{aligned} \tag{13}$$

Using (13) in (12), we find that

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} |e(s)|^2 \leqslant K \int_0^t \mathbb{E}|e(s)|^2 \,\mathrm{d}s + K\Delta t \int_0^t \mathbb{E}(1 + |Y(s^-)|^q + |\overline{Y}(s^-)|^q) \,\mathrm{d}s + \mathbb{E} \sup_{0\leqslant s\leqslant t} |M(s)|. \tag{14}$$

Now, as in the proof of [7, Theorem 4.4], the Burkholder–Davis–Gundy inequality can be used to get the estimate

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} |M(s)| \leqslant \frac{1}{2}\mathbb{E} \sup_{0\leqslant s\leqslant t} |e(s)|^2 + K \int_0^t \mathbb{E}|e(s)|^2 \,\mathrm{d}s + K\Delta t. \tag{15}$$

Using this in (14), along with the moment bounds for $Y(t)$ and $\overline{Y}(t)$, we obtain

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} |e(s)|^2 \leqslant K \int_0^t \mathbb{E} \sup_{0\leqslant r\leqslant s} |e(r)|^2 \,\mathrm{d}s + K\Delta t,$$

and the result now follows from the Gronwall inequality. □

4. Split step backward Euler

In [5], the *split-step backward Euler* (SSBE) method for (1) was defined by $Y_0 = X_0$ and

$$Y_n^\star = Y_n + f(Y_n^\star)\Delta t, \tag{16}$$

$$Y_{n+1} = Y_n^\star + g(Y_n^\star)\Delta W_n + h(Y_n^\star)\Delta N_n, \tag{17}$$

with corresponding continuous-time approximation $\overline{Y}(t)$ defined by (9) and (10). The intermediate approximation $Y_n^\star$ requires a nonlinear equation to be solved, and in [5] it is explained that under the one-sided Lipschitz condition (4), a unique solution is guaranteed, with probability one, for all $\Delta t\mu < 1$. Then defining $F_{\Delta t} : \mathbb{R}^n \to \mathbb{R}^n$ by $F_{\Delta t}(x) = y$, where y uniquely satisfies $y = x + f(y)\Delta t$, it follows that SSBE in (16)–(17) is equivalent to the explicit Euler–Maruyama method (8) applied to the SDE

$$\mathrm{d}X_{\Delta t}(t) = f_{\Delta t}(X_{\Delta t}(t^-))\,\mathrm{d}t + g_{\Delta t}(X_{\Delta t}(t^-))\,\mathrm{d}W(t) + h_{\Delta t}(X_{\Delta t}(t^-))\,\mathrm{d}N(t), \tag{18}$$

with $X(0^-) = X_0$, where

$$f_{\Delta t}(x) = f(F_{\Delta t}(x)), \quad g_{\Delta t}(x) = g(F_{\Delta t}(x)), \quad h_{\Delta t}(x) = h(F_{\Delta t}(x)).$$

Following [7, Lemma 3.4 and Lemma 4.5] it may be shown that $f_{\Delta t}(x)$, $g_{\Delta t}(x)$ and $h_{\Delta t}(x)$ satisfy analogous conditions to $f(x)$, $g(x)$ and $h(x)$, that is, (3)–(7), with possibly larger constants, and, also, for some constant c' and positive integer q'

$$\max\{|f(a) - f_{\Delta t}(a)|^2, |g(a) - g_{\Delta t}(a)|^2, |h(a) - h_{\Delta t}(a)|^2\} \leqslant c'(1 + |a|^{q'})\Delta t^2. \tag{19}$$

We may now compare solutions of (1) and (18).

Lemma 1. *Under assumptions* (2)–(7), *the solutions* $X(t)$ *in* (1) *and* $X_{\Delta t}(t)$ *in* (18) *satisfy*

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} |X_{\Delta t}(t) - X(t)|^2 = \mathrm{O}(\Delta t^2).$$

Proof. The proof follows that of [7, Lemma 4.6]. Applying the Itô formula to $|e(t)|^2$, where $e(t) := X(t) - X_{\Delta t}(t)$, we have

$$\begin{aligned} |e(t)|^2 = & \int_0^t 2\langle f(X(s^-)) - f_{\Delta t}(X_{\Delta t}(s^-)), e(s^-)\rangle + |g(X(s^-)) - g_{\Delta t}(X_{\Delta t}(s^-))|^2\,\mathrm{d}s \\ & + \int_0^t 2\langle e(s^-), (g(X(s^-)) - g_{\Delta t}(X_{\Delta t}(s^-)))\,\mathrm{d}W(s)\rangle \\ & + \int_0^t 2\langle e(s^-), h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\rangle \\ & + |h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))|^2\,\mathrm{d}N(s). \end{aligned} \tag{20}$$

Now, using the one-sided Lipschitz condition (4) along with the Cauchy–Schwarz inequality and the growth bound (19) for *f*, we have

$$\begin{aligned} \int_0^t 2\langle f(X(s^-)) - f_{\Delta t}(X_{\Delta t}(s^-)), e(s^-)\rangle\,\mathrm{d}s = & \int_0^t 2\langle f(X(s^-)) - f(X_{\Delta t}(s^-)), e(s^-)\rangle \mathrm{d}s \\ & + \int_0^t 2\langle f(X_{\Delta t}(s^-)) - f(X_{\Delta t}(s^-)), e(s^-)\rangle\,\mathrm{d}s \\ \leqslant & K \int_0^t |e(s^-)|^2\,\mathrm{d}s + K\Delta t^2 \int_0^t 1 + |X_{\Delta t}(s^-)|^2\,\mathrm{d}s. \end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned} \int_0^t 2|g(X(s^-)) - g_{\Delta t}(X_{\Delta t}(s^-))|^2\,\mathrm{d}s \leqslant & 2\int_0^t 2|g(X(s^-)) - g(X_{\Delta t}(s^-))|^2 + |g(X_{\Delta t}(s^-)) - g_{\Delta t}(X_{\Delta t}(s^-))|^2\,\mathrm{d}s \\ \leqslant & K \int_0^t |e(s^-)|^2\,\mathrm{d}s + K\Delta t^2 \int_0^t 1 + |X_{\Delta t}(s^-)|^2\,\mathrm{d}s. \end{aligned} \tag{22}$$

Now, using the compensated Poisson process (11),

$$\int_0^t 2\langle e(s^-), h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\rangle + |h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))|^2 \,\mathrm{d}N(s)$$
$$= \int_0^t 2\langle e(s^-), h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\rangle + |h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))|^2 \,\mathrm{d}\widetilde{N}(s)$$
$$+ \lambda \int_0^t 2\langle e(s^-), h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))\rangle + |h(X(s^-)) - h_{\Delta t}(X_{\Delta t}(s^-))|^2 \,\mathrm{d}s. \tag{23}$$

The deterministic integrals in (23) can be handled by the approach that led to (21) and (22), so that, overall, from (20) we have

$$|e(t)|^2 \leqslant K \int_0^t |e(s^-)|^2 \,\mathrm{d}s + K\Delta t^2 \int_0^t 1 + |X_{\Delta t}(s^-)|^2 \,\mathrm{d}s + M(t), \tag{24}$$

where $M(t)$ is a martingale that, in the same way as in the proof of Theorem 1, satisfies (15). Using this in (24) and applying the Gronwall inequality completes the proof. □

Because of the connection between SSBE and Euler, Lemma 1 combines with Theorem 1 to give a convergence result for SSBE.

Theorem 2. *Under assumptions* (2)–(7), *the continuous-time extension* (9) *of the SSBE method* (16)–(17) *satisfies*

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} |\overline{Y}(t) - X(t)|^2 = \mathrm{O}(\Delta t).$$

Proof. SSBE is equivalent to the Euler–Maruyama method applied to the modified problem (18). From [5, Lemma 4], we know that $\overline{Y}(t)$ has bounded moments. Hence, from Theorem 1,

$$\mathbb{E} \sup_{0\leqslant t\leqslant T} |\overline{Y}(t) - X_{\Delta t}(t)|^2 = \mathrm{O}(\Delta t).$$

Lemma 1and the triangle inequality complete the proof. □

A variation of SSBE that discretises the compensated version of the jump SDE was also considered in [5]. This *compensated split-step backward Euler* (CSSBE) method for (1) is defined by $Y_0 = X_0$ and

$$Y_n^\star = Y_n + (f(Y_n^\star) + \lambda h(Y_n^\star))\Delta t, \tag{25}$$

$$Y_{n+1} = Y_n^\star + g(Y_n^\star)\Delta W_n + h(Y_n^\star)\Delta\widetilde{N}_n, \tag{26}$$

where $\Delta\widetilde{N}_n := \widetilde{N}(t_{n+1}) - \widetilde{N}(t_n)$. Compared with SSBE this method was shown to require a slightly more stringent restriction on the stepsize to guarantee existence and uniqueness under the one-sided Lipschitz condition (4), but to offer superior linear and nonlinear stability properties, including natural analogues of A- and B-stability. The analysis leading to Theorem 2 can be adapted straightforwardly to show that CSSBE also converges with strong order $\frac{1}{2}$ under the same conditions.

5. Backward Euler

Perhaps the most natural extension of the deterministic backward Euler method to the jump-SDE (1) is given by $Z_0 = X_0$ and

$$Z_{n+1} = Z_n + \Delta t f(Z_{n+1}) + g(Z_n)\Delta W_n + h(Z_n)\Delta N_n. \tag{27}$$

Under the one-sided Lipschitz condition (4), this implicit method has the same existence and uniqueness properties as SSBE. We now show that it also shares the same strong convergence order under the conditions of Theorem 2. The proof exploits a connection between the two methods.

Theorem 3. *Under assumptions* (2)–(7), *there exists a continuous-time extension* $\overline{Z}(t)$ *of the backward Euler method* (27) *that satisfies*

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} |\overline{Z}(t) - X(t)|^2 = \mathrm{O}(\Delta t).$$

Proof. Our proof is a generalisation of the proof of [7, Theorem 5.3]. First, let $\widehat{X}_{\Delta t}(t)$ denote the solution to the SDE (1) with initial data $X_0 - \Delta t f(X_0)$. Then by applying the Itô lemma to $|X(t) - \widehat{X}_{\Delta t}(t)|^2$ it can be shown that this perturbation to the initial data has a controllable effect:

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} |X(t) - \widehat{X}_{\Delta t}(t)|^2 = \mathrm{O}(\Delta t). \tag{28}$$

Now, letting Y_n denote the SSBE approximation for (1) with initial data $Y_0 = X_0 - \Delta t f(X_0)$, it follows by construction that $\{Y_k\}_{k \geqslant 0}$ and $\{Z_k\}_{k \geqslant 0}$ are related by

$$Z_k = Y_k + \Delta t f_{\Delta t}(Y_k).$$

Hence, letting $\overline{Z}(t)$ and $\overline{Y}(t)$ denote the corresponding continuous-time extensions of $\{Y_k\}_{k \geqslant 0}$ and $\{Z_k\}_{k \geqslant 0}$, respectively, as generated by (9), we have

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} |\overline{Y}(t) - \overline{Z}(t)|^2 \leqslant \Delta t^2 \mathbb{E} \sup_{0 \leqslant t \leqslant T} |f_{\Delta t}(\overline{Y}(t))|^2 = \mathrm{O}(\Delta t^2). \tag{29}$$

Here, we have used the facts that $f_{\Delta t}(\cdot)$ is polynomially bounded and that $\overline{Y}(t)$ has bounded moments.

Now from Theorem 2, we know that SSBE converges with strong order $\frac{1}{2}$; that is,

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} |\overline{Y}(t) - \widehat{X}_{\Delta t}(t)|^2 = \mathrm{O}(\Delta t). \tag{30}$$

We may now combine (28)–(30), using the triangle inequality, to give the result. □

6. Numerical experiment

We finish with a numerical example. We note that it is not trivial to infer computationally a precise strong order of convergence on a nonlinear SDE with no explicit solution available—this underlines the importance of rigorous error analysis.

We took $f(x) = x - x^3$, $g(x) = 1 + x$, $h(x) = 1 + x$ and $X_0 = 1$ (constant). Note that (2)–(7) are satisfied. We set $\lambda = 4$ for the Poisson process intensity and solved over $0 \leqslant t \leqslant T = 1$, giving an average of $\lambda T = 4$ jumps per path. The backward Euler method (27) was used. In this case, a cubic polynomial must be solved at each timestep—we took Z_{n+1} to be the real root closest to Z_n. The Poisson increment over a time interval of length δt was computed using a method detailed in [4]. Letting rand denote a call to a uniform (0, 1) pseudo-random number generator, a pseudocode description of this method is:

```
p = e^{-λδt}
f = p
dN = 0
u = rand
while u > f
        dN = dN + 1
        p = pλδt/dN
        f = f + p
end while
```

To assess the strong error, we first computed $M = 10^3$ Brownian and Poisson paths at a resolution of $\delta t = 2^{-14}$. For each path, we applied the backward Euler method with stepsizes of $\Delta t = \delta t, 2\delta t, 4\delta t, 8\delta t, 16\delta t, 32\delta t$. We let $Z_T^{\Delta t}$ denote the $T = 1$ numerical approximation using a stepsize of Δt, and we note that Theorem 3 implies that

$$\mathbb{E}|Z_T^{\Delta t} - X(T)| \leqslant C \Delta t^{1/2}, \tag{31}$$

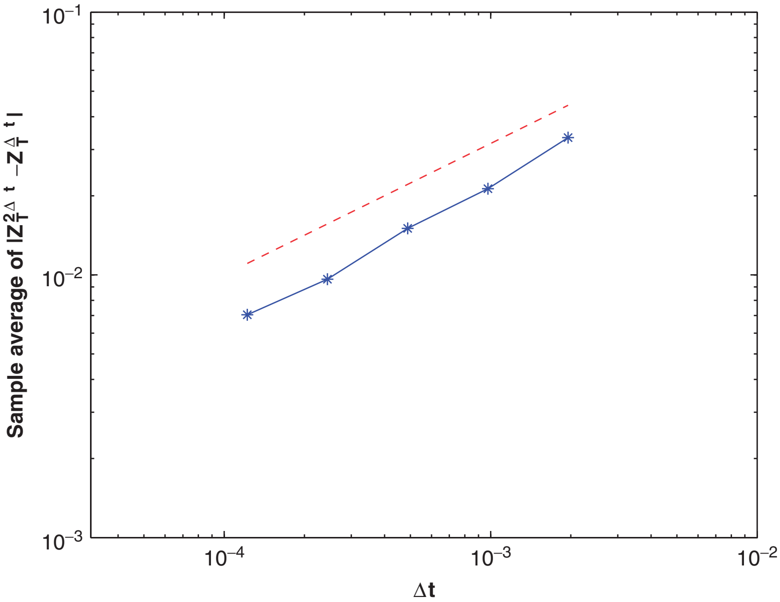

Fig. 1. Asterisks: strong error measure on the left-hand side of (32) for backward Euler applied to a nonlinear jump-SDE. Dashed line: reference slope of $\frac{1}{2}$.

for sufficiently small Δt and some constant C. From the triangle inequality we have

$$|Z_T^{2\Delta t} - Z_T^{\Delta t}| \leqslant |Z_T^{2\Delta t} - X(T)| + |Z_T^{\Delta t} - X(T)|.$$

Taking expected values and using (31) we have

$$\mathbb{E}|Z_T^{2\Delta t} - Z_T^{\Delta t}| \leqslant C(1+\sqrt{2})\Delta t^{1/2}. \tag{32}$$

Fig. 1 shows a log–log plot of the sample mean approximation to $\mathbb{E}|Z_T^{2\Delta t} - Z_T^{\Delta t}|$, based on the M paths, against Δt. A reference line of slope $\frac{1}{2}$ is added in a dashed line type. In this plot, the maximum standard error (that is, the standard deviation divided by $\sqrt{M}$) over all expected value estimates is 1.1×10^{-3}, so the error bars are smaller than the graphics symbols. We see that the computational results are consistent with the bound (32).

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